TRINITY COLLEGE

Mathematics Specialist Units 3 & 4
Test 2 2016

Section 1 Calculator Free

Functions and Sketching Graphs

STUDENT'S NAME: SOLUTIONS

DATE: Thursday 10th March **TIME:** 20 minutes **MARKS**: 23

INSTRUCTIONS:

Standard Items: Pens, pencils, pencil sharper, eraser, correction fluid/tape, ruler, highlighters, Formula Sheet.

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (23 marks)

For the function $f(x) = \dfrac{x^2 - x + 1}{x - 1}$

(a) Determine $f(0)$. $f(0) = -1$ ✓ [1]

(b) State the domain of the function. [1]

$x \in \mathbb{R},\ x \neq 1$ ✓

(c) Determine the real roots (zeros) for the equation $f(x) = 0$. [2]

Consider $x^2 - x + 1 = 0$

$\Delta = (-1)^2 - 4(1)(1) < 0$ ✓

$\therefore$ No real roots ✓

(d) Determine the coordinates and nature (max or min) of any turning points. [4]

$$f'(x) = \frac{(2x-1)(x-1) - (x^2-x+1)(1)}{(x-1)^2}$$ ✓

$f'(x) = 0$ when $(2x-1)(x-1) - (x^2-x+1)(1) = 0$

$\Rightarrow 2x^2 - 3x + 1 - x^2 + x - 1 = 0$

$\Rightarrow x^2 - 2x = 0$ ✓

$\Rightarrow x(x-2) = 0$

$\therefore x = 0$ or $x = 2$

$\therefore (0, -1)$ Local Max. ✓ and $(2, 3)$ Local Min. ✓

(e) State any asymptotes for the function. [3]

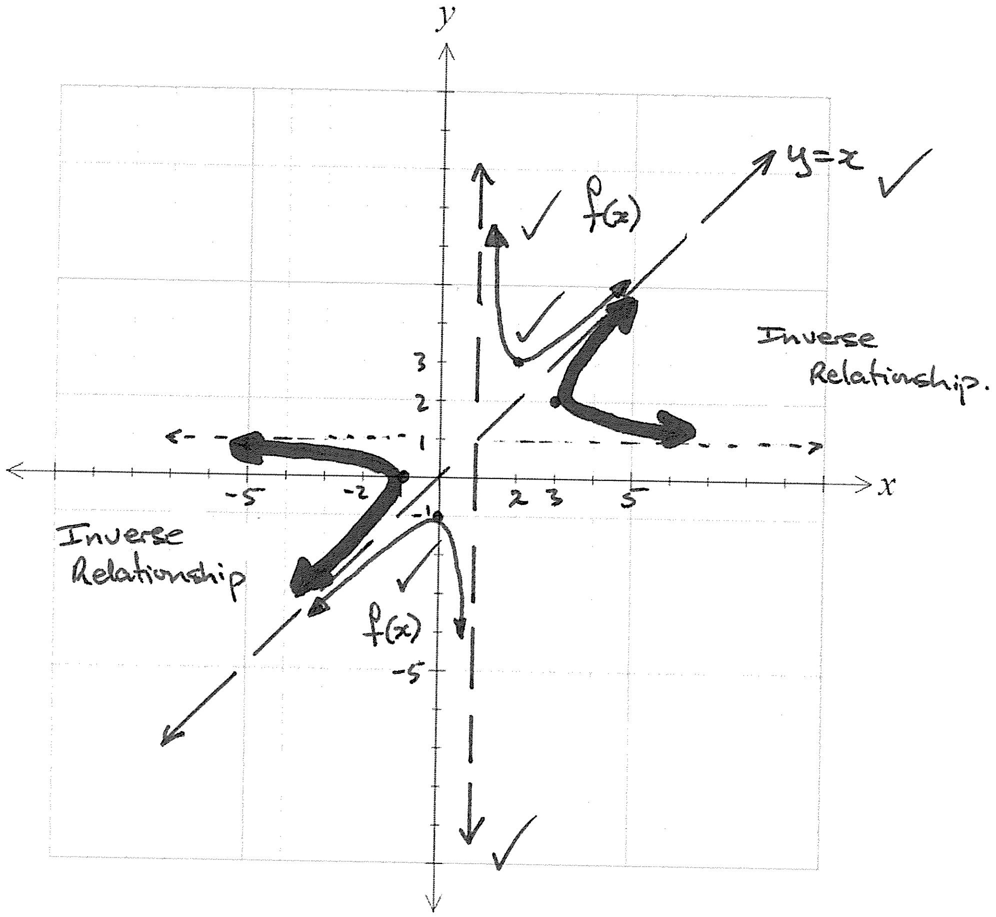
Vertical asymptote (pole) at $x = 1$ ✓

$$\begin{array}{r} x \\ x-1 \overline{) x^2 - x + 1} \\ x^2 - x \\ \hline +1 \end{array}$$

$\therefore f(x) = x + \dfrac{1}{x-1}$ ✓

i.e. Oblique asymptote $y = x$ ✓

(f) Sketch the graph of the function, clearly labelling all the above features. [5]



(g) State the range of the function. [2]

$y \in \mathbb{R};\quad y \leq -1 \quad \text{or} \quad y \geq 3$ ✓ ✓

(h) What type of relationship is this function? [1]

m-1 ✓

(i) Graph the inverse relationship on the same set of axes above. [2]

see above ✓✓

(j) Does $f^{-1}(x)$ exist? If so, why? If not, why not? [2]

$f^{-1}(x)$, the inverse function, does not exist. ✓
as a 1-m relationship
is not a function ✓

End of Questions

TRINITY COLLEGE

Mathematics Specialist Units 3 & 4
Test 2 2016

Section 2 Calculator Assumed

Don't forget you have a calculator.

Functions and Sketching Graphs

STUDENT'S NAME: ______________________________

DATE: Thursday 10th March **TIME:** 25 minutes **MARKS**: 27

INSTRUCTIONS:

Standard Items: Pens, pencils, pencil sharper, eraser, correction fluid/tape, ruler, highlighters, Formula Sheet retained from Section 1.

Special Items: Drawing instruments, templates, three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment).

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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2. (7 marks)

If $f(x)=\dfrac{x}{1-\sqrt{x}}$ and $g(x)=9-2x^2$, determine:

(a) The domain and range for $f(x)$. [4]

Domain: $x \geq 0$, $x \neq 1$; $x \in \mathbb{R}$. ✓ ✓

Range: $y \leq -4$ or $y \geq 0$; $y \in \mathbb{R}$. ✓ ✓

View on Calculator.

(b) State the necessary minimum restriction on the natural domain of $g(x)$ so that $y = f(g(x))$ exists. [3]

The range (output) of $g(x)$ needs to be: $\underline{0 \leq y} \leq 9$, $\underline{y \neq 1}$ ✓ to input to $f(x)$ (Domain)

Consider: $9-2x^2=0$ $\Rightarrow x = \pm\sqrt{4.5}$

$9-2x^2 \neq 1$ $\Rightarrow x \neq \pm 2$

$\therefore \underline{-\sqrt{4.5} \leq x \leq \sqrt{4.5}}$, $\underline{x \neq \pm 2}$; $\underline{x \in \mathbb{R}}$ ✓ ✓

3. (5 marks)

For the function $f(x)=\left|\frac{2x-1}{x-3}\right|$ where $\frac{1}{2}\le x<3$, determine the inverse function $f^{-1}(x)$.

Given $\frac{1}{2}\le x<3 \Rightarrow f(x)=y=-\left(\frac{2x-1}{x-3}\right)$ ✓

Interchange x with y

$\Rightarrow x=-\left(\frac{2y-1}{y-3}\right)$ ✓

$\Rightarrow x(y-3)=-(2y-1)$

$\Rightarrow xy-3x=-2y+1$ ✓

$\Rightarrow xy+2y=3x+1$

$\Rightarrow y(x+2)=3x+1$

$\Rightarrow y=\frac{3x+1}{x+2}$ ✓

$\therefore f^{-1}(x)=\frac{3x+1}{x+2}$ ✓ #

4. (5 marks)

Given that $f(g(x))=\frac{2}{1-x}$ and $f(x)=\frac{x}{x+1}$, determine the rule for $g(x)$.

Given: $f(g(x))=\frac{g(x)}{g(x)+1}$ ✓ $=\frac{2}{1-x}$ ✓

$\Rightarrow g(x)(1-x)=2(g(x)+1)$ ✓

$\Rightarrow g(x)-x\,g(x)=2g(x)+2$

$\Rightarrow -2=g(x)+x\,g(x)$

$\Rightarrow -2=g(x)(1+x)$ ✓

$\therefore g(x)=\frac{-2}{x+1}$ ✓ #

Alternative method:

$g(x)=f^{-1}(f(g(x)))$

Let $y=f(x)$

$\Rightarrow y=\frac{x}{x+1}$

Interchange x with y

$\Rightarrow x=\frac{y}{y+1}$

$\Rightarrow y=\frac{x}{1-x}$

$\therefore f^{-1}(x)=\frac{x}{1-x}$

$\Rightarrow g(x)=f^{-1}\left(\frac{2}{1-x}\right)=\frac{\frac{2}{1-x}}{1-\frac{2}{1-x}}=\underline{\underline{\frac{-2}{x+1}}}$ as above.

5. (10 marks)

* We mathematicians say: $\lim_{x \to 1} f(x) = \frac{4}{3}$

The graph below is a pretty good, but not a perfect, representation of the function:

As $x \to 1$, $f(x) \to \frac{4}{3}$ *

$$f(x) = \frac{x^2+2x-3}{x^2+x-2} = \frac{(x+3)\cancel{(x-1)}}{(x+2)\cancel{(x-1)}}$$

$x \neq 1$ Finite discontinuity (hole) at $(1, \frac{4}{3})$

$x \neq -2$ Pole at $x = -2$

(a) Clearly adjust the graph to improve the representation. See below [2]

(b) On the same set of axes below sketch and label the graphs of:

(i) $y = \dfrac{1}{f(x)}$ [4]

Beware 'holes' and poles; Horiz. asym.

✓✓ @ $x = -2$ and 1 ✓ @ $x = -3$ ✓

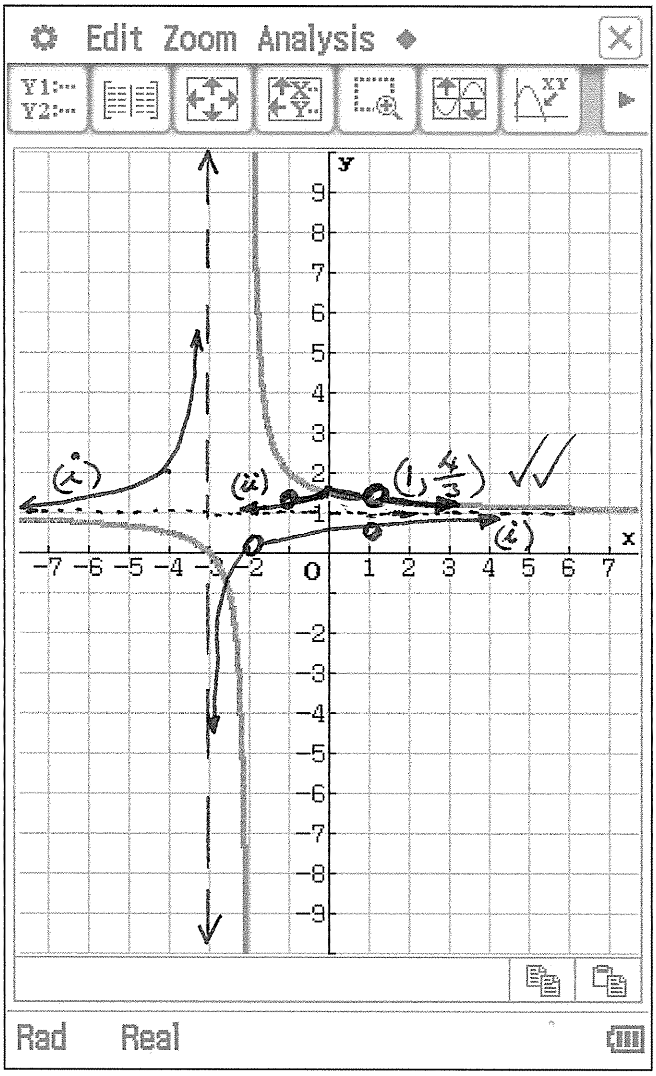
See Below

(ii) $y = f(|x|)$ [4]

Beware 'holes' @ $x = -1$ and 1 ✓ ✓

Horiz. asym. ✓

Shape at $(0, \frac{3}{2})$ ✓ "a cusp"



End of Questions